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Elliptic PDEs, Measures and Capacities

From the Poisson Equation to Nonlinear Thomas–Fermi Problems

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